ON VISIBILITY PROBLEMS WITH AN INFINITE DISCRETE, SET OF OBSTACLES

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Abstract. This paper deals with visibility problems in Euclidean spaces where the set of obstacles Y is an infinite discrete point set. We prove five independent results.

Consider the following problem. Given $\varepsilon > 0$, imagine a forest whose trees have radius ε and their locations are given by the set Y. Suppose that a light source is at infinity, and that there are no arbitrarily large clearings in the forest. Then, are there always dark points (namely, points that do not see infinity)? We answer the above question positively. We also examine other visibility problems. In particular we show that there exists a relatively dense subset Y of $\mathbb{Z}^d$ such that every point in $\mathbb{R}^d$ has a ray to infinity with positive distance from Y.

In addition, we derive a number of other results clarifying how the sizes of the sets of obstacles may affect the sets of points that are visible from infinity. We also present a geometric Ramsey type result concerning finding patterns in uniformly separated subsets of the plane, whose growth is faster than linear.

1. Introduction

We use the following standard notations for a fixed integer $d \geq 2$.

For $x \in \mathbb{R}^d$, we write $\|x\|$ to denote the Euclidean norm of x. Denote by $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d \mid \|x\| = 1\}$ the unit sphere centered at the origin $\mathbf{0}$. For two non-empty subsets $A, B \subseteq \mathbb{R}^d$, define

$$\mathbf{dist}(A, B) = \inf\{\|a - b\| \colon a \in A,\ b \in B\}. \tag{1.1}$$

Given $x \in \mathbb{R}^d$ and $v \in \mathbb{S}^{d-1}$, *by the ray from x in direction v* we mean the set

$$L_{x,v} = \{x + tv \mid t \in [0, \infty)\} \subseteq \mathbb{R}^d.$$

Definition 1.1. For a non-empty subset $Y \subseteq \mathbb{R}^d$, a direction $v \in \mathbb{S}^{d-1}$ and $\varepsilon > 0$, define the following subsets of $\mathbb{R}^d$:

$$\mathbf{vis}(Y,v) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d \mid \mathbf{dist}(L_{x,v}, Y\backslash\{x\}) > 0\}, \tag{1.2a}$$

$$\mathbf{vis}(Y) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d \mid x \in \mathbf{vis}(Y,v) \text{ for some } v \in \mathbb{S}^{d-1}\}, \tag{1.2b}$$

$$\mathbf{vis}(Y,v,\varepsilon) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d \mid \mathbf{dist}(L_{x,v}, Y\backslash\{x\}) \geq \varepsilon\}, \tag{1.2c}$$

$$\mathbf{vis}(Y;\varepsilon) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d \mid x \in \mathbf{vis}(Y,v,\varepsilon) \text{ for some } v \in \mathbb{S}^{d-1}\}. \tag{1.2d}$$

If there is no ambiguity regarding the choice of $Y \subseteq \mathbb{R}^d$, then

- Points $x \in \mathbf{vis}(Y,v)$ are called *visible from direction* v.
- Points $x \in \mathbf{vis}(Y)$ are called *visible*; $x \in \mathbb{R}^d\backslash\mathbf{vis}(Y)$ are called *hidden*.
- Points $x \in \mathbf{vis}(Y,v,\varepsilon)$ are called ε*-visible from direction* v.
- Points $x \in \mathbf{vis}(Y;\varepsilon)$ are called ε*-visible*; $x \in \mathbb{R}^d \setminus \mathbf{vis}(Y;\varepsilon)$ are called ε*-hidden*.

We add the specification "for Y" at the end of the above word definitions if we want to indicate the dependence of these sets on Y.

For $x \in \mathbb{R}^d$ and $r > 0$, denote by $B(x,r) = \{y \in \mathbb{R}^d \mid \|y - x\| < r\}$ the open ball of radius r centered at x. A set $Y \subseteq \mathbb{R}^d$ is called *discrete* if the intersection of Y with every ball $B(x,r)$ is finite.

By the growth rate of a discrete subset $Y \subseteq \mathbb{R}^d$ we mean the integer valued function defined by the formula

$$G_Y(r) = \#\big(\{y \in Y \mid \|y\| < r\}\big) = \#\big(Y \cap B(\mathbf{0},r)\big) \qquad (\text{for } \ r \geq 0).$$

(Here and henceforth $\#S$ stands for the cardinality of a set S).

A set $Y \subseteq \mathbb{R}^d$ is called *relatively dense* if there exists an $r > 0$ so that Y intersects every ball of radius r. Y is *uniformly separated* if there exists a $\delta > 0$ such that for every $y_1, y_2 \in Y$ we have $\mathbf{dist}(y_1, y_2) \geq \delta$. We say Y is r-dense, or δ-separated, when we like to specify the constants r and δ. Note that if $Y \subseteq \mathbb{R}^d$ is a uniformly separated set then $\limsup_{r\to\infty} \left|\frac{G_Y(r)}{r^d}\right| < \infty$, and if $Y \subseteq \mathbb{R}^d$ is discrete and relatively dense then $\limsup_{r\to\infty} \left|\frac{r^d}{G_Y(r)}\right| < \infty$.

One of the objectives in this paper is to investigate how the growth rate of the set Y may be related to certain properties of the four sets defined in (1.2). Note that $Y_1 \subseteq Y_2$ implies the inequality $G_{Y_1}(r) \leq$

$G_{Y_2}(r)$ (for all $r > 0$) and the inclusions

$$\mathbf{vis}(Y_2, v) \subseteq \mathbf{vis}(Y_1, v), \qquad \mathbf{vis}(Y_2) \subseteq \mathbf{vis}(Y_1), \tag{1.3}$$

$$\mathbf{vis}(Y_2; \varepsilon) \subseteq \mathbf{vis}(Y_1; \varepsilon), \qquad \mathbf{vis}(Y_2, v, \varepsilon) \subseteq \mathbf{vis}(Y_1, v, \varepsilon). \tag{1.4}$$

In particular, we establish the following results for discrete subsets $Y \subseteq \mathbb{R}^d$:

1. If $Y \subseteq \mathbb{R}^2$ is relatively dense then for all $\varepsilon > 0$ we have $\mathbf{vis}(Y; \varepsilon) \neq \mathbb{R}^2$ (See Theorem 1.2).
2. On the other hand, there exists a relatively dense set $Y \subseteq \mathbb{R}^d$ ($d \geq 2$) such that $\mathbf{vis}(Y) = \mathbb{R}^d$ (See Theorem 1.5).
3. If $Y \subseteq \mathbb{R}^d$ is discrete with $G_Y(t) < \frac{t^{d-1}}{\log^{1+\varepsilon} t}$, for some $\varepsilon > 0$ and all large enough t, then $\mathbf{vis}(Y) = \mathbb{R}^d$ (See Theorem 1.4).
4. On the other hand, we exhibit an uniformly separated set $Y \subseteq \mathbb{R}^2$ such that $\lim\limits_{t\to\infty} \frac{G_Y(t)\log t}{t} = 1$ and $\mathbf{vis}(Y) = \varnothing$ (See Theorem 1.3).

The visibility notions from Definition 1.1 relate to the well-known Pólya's orchard problem (see [12, 13]): What is the minimal radius of trees (viewed as disks in $\mathbb{R}^2$), that stand at the integer points in a ball of radius R, for them to completely block the visibility of the origin, from the boundary of the ball? this problem was solved by Allen in [2], and some variants of it appear in [8, 9]. One may also consider a maximal packing of unit balls in a ball of radius R, instead of balls at integer points, and ask for which R (if any) there exists points that are not visible from the boundary? The existence of such an R is known as Mitchell's dark forest conjecture, see [10]. Mitchell's conjecture was proved in [6]. Another related notion is the following, which can be viewed as a quantified version of a point set for which every point in $\mathbb{R}^d$ is hidden. $Y \subseteq \mathbb{R}^d$ is called a *dense forest* if for every $\varepsilon > 0$ there is a uniform upper bound $T(\varepsilon) > 0$ on the lengths of the line segments that are not ε-close to Y. $T(\varepsilon)$ is called the visibility function of Y. Questions regarding the existence of dense forests that are uniformly separated, or of bounded density, and bounds on the visibility functions of them, were studied in [1, 3, 4, 14].

Our main results are the following five theorems; Theorems 1.2, 1.3, 1.4, 1.5 and 1.7.

Theorem 1.2. *For every R and ε, where $R \geq \varepsilon > 0$, and for every R-dense set $Y \subseteq \mathbb{R}^2$, there exist some $T > 0$ with the following property: for every $x_0 \in \mathbb{R}^2$ there are points $x \in B(x_0, T) \smallsetminus \bigcup_{y\in Y} B(y, \varepsilon)$ such that for every $v \in \mathbb{S}^1$ we have*

$$\mathbf{dist}(Y, \{x + tv \mid t \in [0, T]\}) < \varepsilon.$$

In particular, we have $\mathbf{vis}(Y;\varepsilon) \neq \mathbb{R}^2$ *and the set of* ε*-hidden points is itself* T*-dense.*

We remark that by rescaling one may assume that one of the two parameters R and ε in Theorem 1.2 is fixed. Fixing $\varepsilon = 1$ for example, would simplify the constants that appear in the proof of Theorem 1.2. In order to present the explicit dependency of T in R and ε, we let both of the parameters vary. It would be interesting to find a smaller T, in terms of R/ε, that satisfies Theorem 1.2.

Theorems 1.3 and 1.4 address the connection between the growth rate of a discrete set Y and properties of the set $\mathbf{vis}(Y)$.

Theorem 1.3. *There exists a uniformly separated set* $Y \subseteq \mathbb{R}^2$ *such that* $\mathbf{vis}(Y) = \varnothing$ *and*

$$\lim_{r\to\infty} \frac{G_Y(r)\log r}{r} = 1.$$

(In particular, the growth rate of such Y *is sublinear,* $\lim\limits_{r\to\infty} \frac{G_Y(r)}{r} = 0$*, and all points in* $\mathbb{R}^2$ *are hidden for* Y*).*

Theorem 1.4. *Let* $Y \subseteq \mathbb{R}^d$ *be a discrete set. Then the implications (1)⇒(2)⇒(3) take place, where*

(1) $G_Y(r) < \frac{r^{d-1}}{\log^{1+\varepsilon} r}$*, for some* $\varepsilon > 0$ *and all large* r*.*

(2) $\sum\limits_{y\in Y\setminus\{\mathbf{0}\}} \frac{1}{\|y\|^{d-1}} < \infty$.

(3) *For every* $x \in \mathbb{R}^d$*, the relation* $x \in \mathbf{vis}(Y, v)$ *holds for Lebesgue almost all* $v \in \mathbb{S}^{d-1}$*. In particular,* $\mathbf{vis}(Y) = \mathbb{R}^d$*.*

It is easy to see that, for all $d \geq 2$, $\mathbf{vis}(\mathbb{Z}^d) = \mathbb{R}^d \setminus \mathbb{Z}^d$. On the other hand, the following theorem shows existence of large (density 1 and relatively dense) subsets $Y \subseteq \mathbb{Z}^d$ with no hidden points for Y.

Theorem 1.5. *Let* $d \geq 2$*. Then, for any* $\varepsilon > 0$ *and* $M > 1$*, there exists a subset* $Y \subseteq \mathbb{Z}^d$ *that has the following properties:*

(1) $\mathbf{vis}(Y) = \mathbb{R}^d$ *(that is, there are no hidden points for* Y*).*
(2) *The growth rate of the complement set* $\tilde{Y} = \mathbb{Z}^d \setminus Y$ *is at most linear; moreover,* $G_{\tilde{Y}}(r) = \#\big(\tilde{Y} \cap B(\mathbf{0}, r)\big) < \varepsilon r$*, for all* $r > 0$*.*
(3) Y *is relatively dense in* $\mathbb{Z}^d$*.*
(4) *The set* $\tilde{Y}$ *is* M*-separated.*

The set Y in Theorem 1.5 can be viewed as a discrete analogue of a *Nikodym set*, a full measure subset N of the unit square in the plane such that for every $x \in N$ there is a line segment L_x satisfying $N \cap L_x = \{x\}$, see [7] and [11].

As an application of our approach we also prove a geometric Ramsey-type theorem, Theorem 1.7 below. Given a discrete set Y we say that *almost every* $y \in Y$ satisfies (some) property (P) if

$$\lim_{r\to\infty} \frac{\#\{y \in Y \cap B(\mathbf{0}, r) |\ y \text{ does not have property } (P)\}}{G_Y(r)} = 0. \quad (1.5)$$

Definition 1.6. Let $Y \subseteq \mathbb{R}^2$ be discrete set, $\varepsilon > 0$, and Γ a tree[1] embedded in the plane with vertices $V = \{x_0, \ldots, x_m\}$. Given $y_0 \in Y$, we say that (Γ, x_0) *can be* ε*-realized from* y_0 *in* Y if there exists a function $f \colon V \to Y$ such that $f(x_0) = y_0$ and for every edge $\{x_i, x_j\}$ of Γ there is an integer $k_{ij} \geq 1$ such that

$$\|(f(x_i) - f(x_j)) - k_{ij}(x_i - x_j)\| < \varepsilon.$$

Theorem 1.7. *Let* $\varepsilon > 0$*,* $Y \subseteq \mathbb{R}^2$ *a uniformly separated set with* $\lim\limits_{T\to\infty} \frac{T}{G_Y(T)} = 0$*,* $\Gamma = (V, E)$ *a finite tree embedded in* $\mathbb{R}^2$*, and* $x_0 \in V$*. Then for almost every* $y_0 \in Y$ *(in the sense of* (1.5)*),* (Γ, x_0) *can be* ε*-realized from* y_0 *in* Y*.*

Figure 1 illustrates the statement of the theorem.

The structure of the paper. The proofs of the Theorems are given in the order they are presented in the introduction. The proofs of Theorems 1.2, 1.3, 1.4 and 1.5 are given in Sections 2, 3, 4 and 5 respectively. The proof of Theorem 1.7 is presented in section 6.

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2. Proof of Theorem 1.2

Theorem 1.2 resembles the following theorem from [6].

Theorem 2.1. *For every* ***maximal*** *packing of the plane* ***by unit balls*** $\{B_i\}$ *there exists a* $T > 0$ *and a point* $x \in B(\mathbf{0}, T) \smallsetminus \bigcup_i B_i$ *so that every line segment between* x *and* $\partial B(\mathbf{0}, T)$ *intersects an element of the collection* $\{B_i\}$*.*

[1] An undirected, acyclic, connected graph.

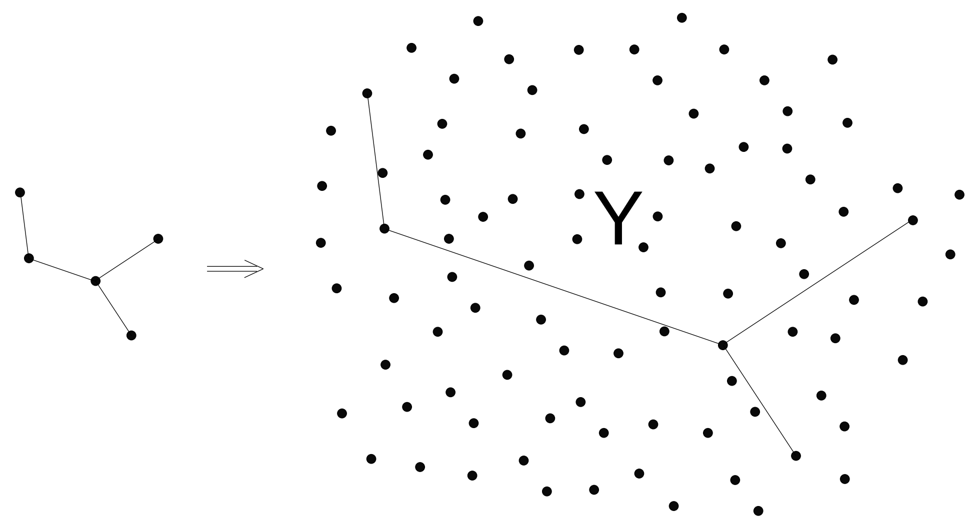

FIGURE 1. Different edges may be stretched by different integer factors.

Note that Theorem 2.1 can be deduced from our Theorem 1.2. Set Y to be the collection of centers of the balls B_i, then the assertion of Theorem 1.2, for this Y and with $\varepsilon = 1$, is Theorem 2.1. Our proof follows the main ideas of the proof of Dumitrescu and Jiang from [6], although many parts had to be adapted to our settings. This is done in §2.3.

Our main innovation in Theorem 1.2, compared to Theorem 2.1, is the 'for all' quantifier on the parameters R and ε. Whereas in Theorem 2.1 a maximal packing is required in order to block the visibility from infinity, in our Theorem 1.2 a much more sparse set, of ε-balls that are R apart, suffices.

Note that since some of the parameters that are used in the proof are very large, and some are very small, some of our figures are drawn with wrong proportions.

2.1. **Proof outline.** For every $z \in Y$ let $C_z = \partial B(z, \varepsilon)$. We show that for many elements $z \in Y$ there are points on C_z that are not ε-visible. As in [6] we distinguish between two types of ε-visible points on C_z; points $p \in C_z$ that are ε-visible by a ray that is almost tangent to C_z at p are called *tangentially visible*, and other ε-visible points on C_z are called *frontally visible*. In Lemma 2.3 we show that every circle[2]

[2]Even every arc of every circle.

of radius ε contains points that are not tangentially visible. Then in Lemma 2.7 we show that for a large enough T only a fraction of the circles C_z that are contained in $B(x_0, T)$, for $x_0 \in \mathbb{R}^2$ and $z \in Y$, contains points that are frontally visible. These two together imply that for a large enough T, some portion of the circles C_z in $B(x_0, T)$, $z \in Y$, contain points that are not ε-visible. In particular, such points exist in every ball $B(x_0, T)$.

2.2. **Terminology.** Given a circle C in the plane and $\sigma, \alpha \in [0, 2\pi)$ we denote by $A(\sigma; \alpha)$ the arc of the circle C that corresponds to the central angle that lies between σ and $\sigma + \alpha$. The function $a : [0, 2\pi) \to C$ maps an angle α to the point on C that is the intersection of C and the ray in direction α from the center of C. For two points $x, y \in \mathbb{R}^2$ we denote by $\overline{xy}$ the line segments that connects x and y.

Definition 2.2. Let $\varepsilon, \delta > 0$ and let $C \subseteq \mathbb{R}^2$ be a circle.

- A point $p \in C$ is called *δ-tangentially-ε-visible* (δ-T-ε-V) if p is ε-visible by a ray $L_{p,v}$ that satisfies:
  (i) $L_{p,v} \cap C = \{p\}$ ($L_{p,v}$ intersects C only at the tangent point).
  (ii) The angle between $L_{p,v}$ and the tangent to C at p is at most δ.
- An arc of C is called *completely* δ-T-ε-V if **every** point on that arc is δ-T-ε-V.
- $p = a(0)$ denotes the point where the tangent to C at p is vertical ($a(\pi)$ also has this property, though here $p = a(0)$). There are two directions in which a ray initiated from p is almost tangent to p, and we distinguish between them in the following way. We say that a ray is *pointing downwards* (respectively *upwards*) to describe rays that point in these two directions, up to a small error. We say that p is δ-T-ε-V *from below* (respectively δ-T-ε-V *from above*) if p is δ-T-ε-V by a ray pointing downwards (respectively upwards), up to an error angle δ at p from the tangent to p. We adapt this terminology to other points $q = a(\alpha)$ on C by rotating the plane so that $q = a(0)$. Note that this terminology will be used in the proof for points that are close to $a(0)$, where the rays truly point almost vertically downwards or almost vertically upwards.
- A point $p \in C$ that is ε-visible but not δ-T-ε-V is called *δ-frontally-ε-visible* (δ-F-ε-V). A ball B is called δ-F-ε-V if **some** point on its boundary is δ-F-ε-V.

2.3. **Proof of Theorem 1.2.** We begin with a lemma that asserts that for relatively dense sets Y, completely δ-T-ε-V arcs do not exist, for $\delta = \delta(\varepsilon)$ small enough.

Lemma 2.3. *Let $Y \subseteq \mathbb{R}^2$ be an R-dense set. Then for every $\varepsilon, \alpha > 0$ there exists $\delta = \delta(\varepsilon, R, \alpha) > 0$ such that for every $x \in \mathbb{R}^2$, every arc of central angle α in $C \stackrel{\text{def}}{=} \partial B(x, \varepsilon)$ is not completely δ-T-ε-V.*

Proof. It suffices to prove the statement for $\alpha \in (0, \pi/6)$. Let $\varepsilon > 0, \alpha \in (0, \pi/6)$. To simplify notations, we may assume that R is an integer multiple of ε (replacing R by some number in $[R, R+\varepsilon)$ entail no loss of generality). Set

$$N \stackrel{\text{def}}{=} \frac{2R}{\varepsilon}, \qquad \beta \stackrel{\text{def}}{=} \frac{\alpha}{4N} \qquad \text{and} \qquad \delta \stackrel{\text{def}}{=} \frac{\beta}{4N} = \frac{\alpha}{16N^2}. \tag{2.1}$$

Let $x \in \mathbb{R}^2$, and $C \stackrel{\text{def}}{=} \partial B(x, \varepsilon)$. Without loss of generality we prove the lemma for the arc $A \stackrel{\text{def}}{=} A(0; \alpha)$ (see §2.2). For contradiction, assume that A is completely δ-T-ε-V.

Divide A into $4N$ arcs, of equal length, with the points $q_i \stackrel{\text{def}}{=} a\left(\frac{i\alpha}{4N}\right)$, for $i \in \{0, \dots, 4N\}$ ($a(\cdot)$ as in §2.2). Each of these sub-arcs has central angle β, and we denote it by $A_i \stackrel{\text{def}}{=} A(q_i; \beta)$, for $i \in \{0, \dots, 4N-1\}$. Similarly, divide each A_i into $4N$ arcs of equal length with the points $p_{i,j} \stackrel{\text{def}}{=} a\left(\frac{i\alpha}{4N} + \frac{j\beta}{4N}\right)$, for $j \in \{0, \dots, 4N\}$. Consider the following two cases:

Case 1: There exists an $i \in \{0, \dots, 4N-1\}$ such that all the points $p_{i,1}, \dots, p_{i,4N-1}$ are δ-T-ε-V from below:
For $j \in \{0, \dots, 4N-1\}$ let L_j be the ray tangent to C at $p_{i,j}$ that points downwards, r_j a ray that indicates that $p_{i,j}$ is δ-T-ε-V from below, and L'_j the ray pointing downwards that intersects C only at $p_{i,j}$ and that create an angle δ at $p_{i,j}$ between L_j and L'_j. Let L_{4N} be the ray tangent to $p_{i,4N} = p_{i+1,0}$ that points downwards. Denote by z the intersection point of L_0 and L_{4N} and let a_0 and a_{4N} be two points on L_0 and L_{4N} respectively such that the triangle with vertices a_0, a_{4N}, z is the minimal isosceles triangle that contains a ball B of radius R (see Figure 2 (a)). Since $\beta < \pi/6$, the legs of that triangle are indeed $\overline{za_0}$ and $\overline{za_{4N}}$, and the base is $I \stackrel{\text{def}}{=} \overline{a_0 a_{4N}}$.

For every $j \in \{0, \dots, 4N-1\}$ let b_j be the intersection point of r_j and I, and a_j the intersection point of L_j and I. Our next goal is to show that the points $\{b_0, \dots, b_{4N-1}\}$ divide I into segments of lengths less than ε. This in turn implies that the rays r_j divide B in a way that every ball of radius ε that is centered in B intersects at least

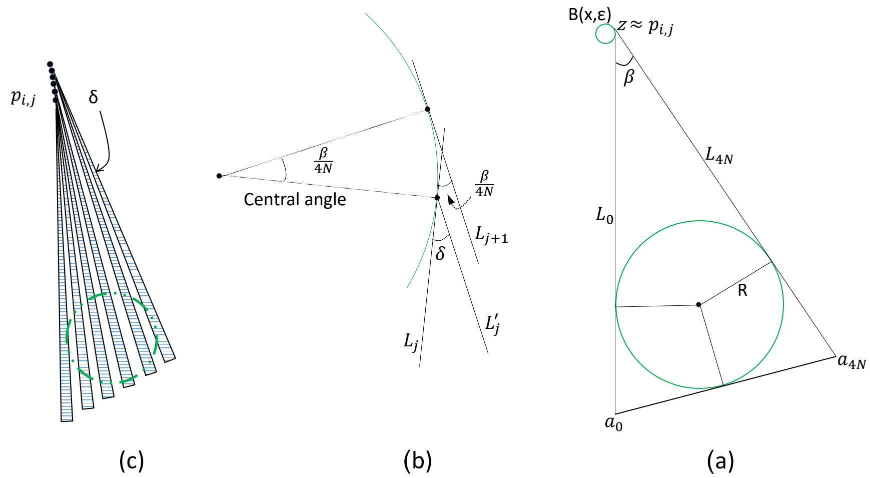

FIGURE 2

one of the rays r_j (see Figure 2 (c)). Since Y is R-dense, there exists some $y \in B \cap Y$. This will contradict the assumption that the points $p_{i,1}, \ldots, p_{i,4N-1}$ are δ-T-ε-V by the rays $r_1, \ldots, r_{4N-1}$ and thus conclude the proof.

Using elementary geometry (see Figure 2 (a)) it is easy to show that for every $j \in \{0, \ldots, 4N-1\}$ we have $\mathbf{dist}(p_{i,j}, a_j) \leq \frac{4R}{\beta}$. Since $\delta = \beta/4N$ the slope of the ray L_{j+1} is equal to the slope of L'_j (see Figure 2 (b)). This implies that b_j lies between a_j and a_{j+1} on I. In addition we have

$$\mathbf{dist}(a_j, a_{j+1}) \leq 2\mathbf{dist}(p_{i,j}, a_j) \sin\left(\frac{\beta}{8N}\right) \leq 2\frac{4R}{\beta}\frac{\beta}{8N} = \frac{R}{N} = \frac{\varepsilon}{2},$$

which implies the assertion.
Case 2: For every $i \in \{0, \ldots, 4N-1\}$ there is a $j \in \{0, \ldots, 4N-1\}$ such that $p'_i \overset{\text{def}}{=} p_{i,j}$ is δ-T-ε-V from above[3]:
We repeat the argument from case 1 in a larger scale. For simplicity, we use the same notations. Here we denote by L_i, for $i \in \{0, \ldots, 4N\}$, the ray tangent to C at $q'_i = a\left(\frac{i\alpha}{4N} - \frac{\beta}{4N}\right)$ that points upwards, and by r_i, for $i \in \{0, \ldots, 4N-1\}$, a ray that indicates that p'_i is δ-T-ε-V from above. Denote by z the intersection point of L_0 and L_{4N} and let a_0 and a_{4N} be two points on L_0 and L_{4N} respectively such that the triangle

[3]Note that this is the opposite case to Case 1.

with vertices a_0, a_{4N}, z is the minimal isosceles triangle that contains a ball B of radius R. Since $\alpha < \pi/6$, the legs of that triangle are $\overline{za_0}$ and $\overline{za_{4N}}$, and the base is $I \stackrel{\text{def}}{=} \overline{a_0 a_{4N}}$.

For $i \in \{0, \ldots, 4N-1\}$ let b_i be the intersection point of r_i and I, and a_i the intersection point of L_i and I. Once again it is easy to verify that $\mathbf{dist}(q_i', a_i) \leq \frac{4R}{\alpha}$, and that b_i lies between[4] a_i and a_{i+1} on I, for every $i \in \{0, \ldots, 4N-1\}$. In addition we have

$$\mathbf{dist}(a_i, a_{i+1}) \leq 2\,\mathbf{dist}(q_i', a_i) \sin\left(\frac{\alpha}{8N}\right) \leq 2\,\frac{4R}{\alpha}\,\frac{\alpha}{8N} = \frac{R}{N} = \frac{\varepsilon}{2},$$

which implies the assertion in a similar manner, and hence completes the proof of Lemma 2.3. □

The proofs of the following two geometric lemmas are straightforward, and we leave them to the reader.

Lemma 2.4. *Let $T > \mu > 0$. Suppose that $a_i = (x_i, y_i) \in \mathbb{R}^2, i \in \{1,2,3,4\}$, satisfy (see Figure 3)*

$$x_1 = x_2 = 5T, y_1, y_2 \in \left[-\frac{\mu}{2}, \frac{\mu}{2}\right], y_1 \leq y_2, x_3 = x_4, y_4 - y_3 \geq \mu, \quad (2.2)$$

and $a_3, a_4 \in B \stackrel{\text{def}}{=} [-T, T]^2$. Let $\ell_1 = \overline{a_1 b}$, $\ell_2 = \overline{a_2 c}$, where $b \neq c$ and $b, c \in \{a_3, a_4\}$. Denote by d_1, d_2 the intersection point of ℓ_1, ℓ_2 respectively with ∂B. Then $\mathbf{dist}(d_1, d_2) \geq \frac{\mu}{3}$ (see Figure 3).

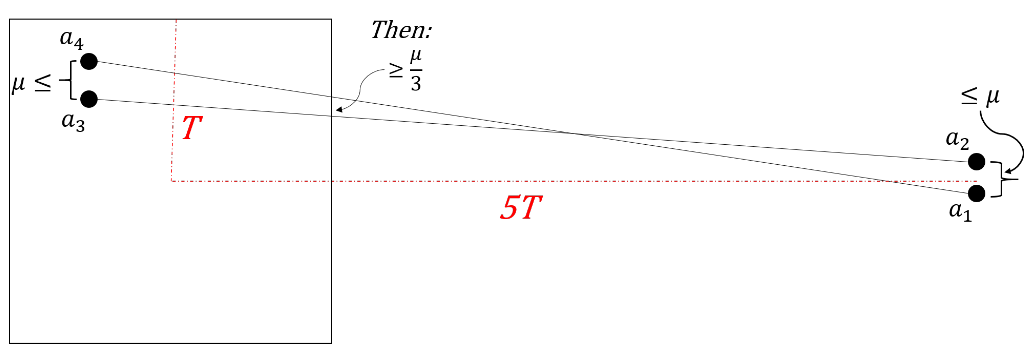

FIGURE 3

[4]This follows from the term $-\frac{\beta}{4N}$ that appears in the definition of the new points q_i'.

Lemma 2.5. *Let $\varepsilon > \delta > 0$. For $z \in \mathbb{R}^2$ if a ball $B(z,\varepsilon)$ is δ-F-ε-V by a ray L then there is a point $a \in B(z,\varepsilon)$, that lies on the continuation of L, with $\mathbf{dist}(a, \partial B(z,\varepsilon)) > \mu$, where*

$$\mu \stackrel{\text{def}}{=} \frac{\varepsilon\delta^2}{4}. \tag{2.3}$$

For the proof of the next lemma we rely on the following proposition, see Lemma 6 in [6].

Proposition 2.6. *Let $k, c, \eta > 0$ and let I be an interval of length $|I|$. Let $A \subseteq I$ be a finite set with at least $c\,|I|$ points, which are at least η apart from each other. Set*

$$r \stackrel{\text{def}}{=} \frac{k}{k-1}, j \stackrel{\text{def}}{=} \left\lceil \frac{\log \frac{2}{c\eta}}{\log r} \right\rceil, \text{ and } Z_0 = Z_0(k,c,\eta) \stackrel{\text{def}}{=} 2\eta k^j. \tag{2.4}$$

Then if $|I| \geq Z_0$ there exists some $x \geq 2\eta$, and a sub-interval $J \subseteq I$ of length kx such that the subdivision of J into k equal sub-intervals $J_1, \ldots, J_k$ satisfies $J_i \cap A \neq \varnothing$ for every i.

Lemma 2.7. *Let $\varepsilon > 0$, let $Y \subseteq \mathbb{R}^2$ be an R-dense set such that R is an integer multiple of ε. Let $N = \frac{2R}{\varepsilon}, \delta = \frac{\varepsilon^2}{2^7 \cdot R^2}$, $C = \frac{1}{4R^2}$, and let $T > 0$ be an integer multiple of $\varepsilon\delta^2$ that satisfies*

$$T \geq \frac{\varepsilon}{2}(4N)^j, \quad \text{where} \quad j = \left\lceil \frac{33 + 10\log N}{\log(4N) - \log(4N-1)} \right\rceil. \tag{2.5}$$

Then the number of $z \in Y \cap B(\mathbf{0}, T)$ such that $B(z,\varepsilon)$ is δ-F-ε-V is less than CT^2.

Proof. For contradiction, assume that for at least CT^2 points $z \in Y \cap B(\mathbf{0}, T)$ the balls $B_z = B(z,\varepsilon)$ are δ-F-ε-V. Each of these balls has a point p_z on its boundary and a ray L_z, initiated at p_z, which indicates that B_z is δ-F-ε-V. Denote by $\mathcal{L}$ the set of the rays L_z, then $\#\mathcal{L} \geq CT^2$.

Set

$$\mu \stackrel{\text{def}}{=} \frac{\varepsilon\delta^2}{4} = \frac{\varepsilon^5}{2^{16}R^4}, \quad \text{and} \quad M \stackrel{\text{def}}{=} \frac{32T}{\mu}. \tag{2.6}$$

Note that the assumption on T implies that M is an integer. Consider the larger ball $B(\mathbf{0}, 5T)$ and place M equally spaced[5] points $p_0, \ldots, p_{M-1}$ on $\partial B(\mathbf{0}, 5T)$. The tangents to $B(\mathbf{0}, 5T)$ through the points p_j form a regular M-gon that $B(\mathbf{0}, 5T)$ is inscribed in. Denote by I_j the edge of that M-gon that contains p_j. Observe that the

[5] Note that $32T > 10\pi T$, which is the length of the diameter of $B(\mathbf{0}, 5T)$.

length of each segment I_j is at most μ. By the pigeonhole principle there exists some j such that at least

$$\frac{CT^2}{M} = \frac{CT^2}{\frac{32T}{\mu}} = \frac{C\mu T}{32} \tag{2.7}$$

rays from $\mathcal{L}$ intersects I_j. We rotate the whole plane about the origin so that the segment I_j is vertical, and denote by $\mathcal{L}_j \subseteq \mathcal{L}$ the subset of rays of $\mathcal{L}$ that intersects I_j. Note that all of these rays intersect the vertical line segment $I \stackrel{\text{def}}{=} \{T\} \times [-T, T]$ of length $2T$ (see Figure 4). Let $A' \subseteq I$ be the set of these intersection points.

Recall that each ray L_z in $\mathcal{L}_j$ is initiated from a point $p_z \in \partial B_z$, for some $z \in Y$, such that p_z is δ-F-ε-V by L_z. So by our choice of μ in (2.6) and by Lemma 2.5 there is a point $a_z \in B_z$ with $\mathbf{dist}(a_z, \partial B_z) \geq \mu$. This implies that the requirements in (2.2) are satisfied (see Figure 3), and we can apply Lemma 2.4 for any such pair of rays, connecting points of the form a_z to I_j (see Figure 4). This in turn implies that the points of A' are at least $\mu/3$ apart from each other, and in particular no two rays of $\mathcal{L}_j$ intersect I at the same point. We pick a subset $A \subseteq A'$ such that any two points in A are at least $\varepsilon/2$ apart. This is done by ordering the elements of A' and picking every $\frac{3\varepsilon}{2\mu}$ point in that order (note that $\frac{3\varepsilon}{2\mu} \in \mathbb{N}$). Thus, using (2.7), we obtain that

$$\frac{\#A}{|I|} \geq \frac{\#A'}{\frac{3\varepsilon}{2\mu}|I|} \geq \frac{\frac{C\mu T}{32}}{\frac{3\varepsilon}{2\mu} \cdot 2T} \geq \frac{C\mu^2}{2^7\varepsilon} = \frac{\mu^2}{2^9 \varepsilon R^2} \stackrel{\text{def}}{=} c. \tag{2.8}$$

We apply Proposition 2.6 with c as in (2.8), $k = 4N$, and $\eta = \varepsilon/2$. In view of (2.6) and (2.8) we obtain

$$c\eta = c\varepsilon/2 = \frac{\mu^2 \varepsilon}{2^{10}\varepsilon R^2} = \frac{\varepsilon^{10}}{2^{42}R^{10}} \quad \Longrightarrow \quad \log\frac{2}{c\eta} = 33 + 10\log\frac{2R}{\varepsilon} = 33 + 10\log N.$$

Therefore the constant j in (2.4) is

$$j = \left\lceil \frac{33 + 10\log N}{\log\frac{4N}{4N-1}} \right\rceil = \left\lceil \frac{33 + 10\log N}{\log(4N) - \log(4N-1)} \right\rceil.$$

Then the constant Z_0 in (2.4) is

$$Z_0 = 2\eta k^j = \varepsilon(4N)^j.$$

Thus, by the assumption on T in (2.5), we have $|I| = 2T \geq Z_0$. Applying Proposition 2.6 we obtain an $x \geq 2\eta = \varepsilon$, and a sub-interval $J \subseteq I$ of length $4Nx \geq 4R$ such that the subdivision of J into $4N$ equal subintervals $J_1, \ldots, J_{4N}$ satisfies $J_i \cap A \neq \varnothing$ for every $i \in \{1, \ldots, 4N\}$. Let $L_1, \ldots, L_{4N} \in \mathcal{L}_j$ be the rays that correspond to those $4N$ points of

A. Let Ω be the convex hull of $I_j \cup J$, then Ω clearly contains balls of radius R. Let $B \subseteq \Omega$ be the ball of radius R that is tangent to the line segments that bound Ω from above and below (see Figure 4). Then $B \cap Y \neq \varnothing$ and every point $p \in B \cap Y$ is within distance at most $\varepsilon/2$ from at least one of the rays $L_1, \dots, L_{4N}$, contradicting our assumption on the rays in $\mathcal{L}$.

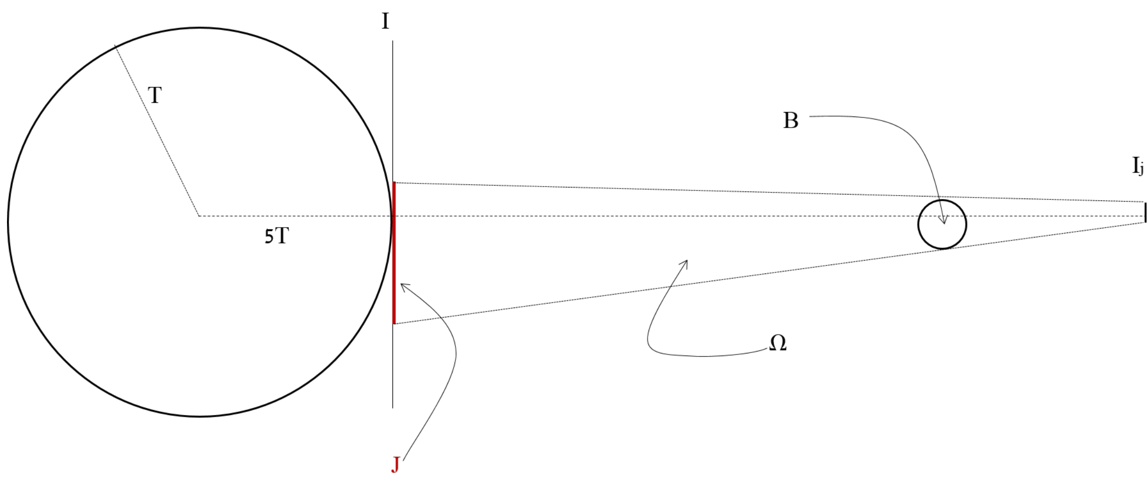

FIGURE 4

□

Proof of Theorem 1.2. Let $\varepsilon > 0$ and let $Y \subseteq \mathbb{R}^2$ be an R-dense set. By slightly increasing R we may assume that R is an integer multiple of ε. Pick $T > 0$ as in Lemma 2.7, and given $x_0 \in \mathbb{R}^2$ we prove that the ball $B \stackrel{\text{def}}{=} B(\mathbf{0}, T)$ contains points of the R-dense set $Y_0 \stackrel{\text{def}}{=} Y - x_0$, with the required property.

It is easy to verify that B contains at least $\frac{1}{2R^2}T^2$ disjoint balls of radius R, each one contains at least one element of Y_0. Denote by $Y' = Y_0 \cap B$, then we have established that $\#Y' \geq \frac{1}{2R^2}T^2$.

Let $\delta = \frac{\varepsilon^2}{2^7 R^2}$, as in Lemma 2.7. Note that this δ is consistent with our choice of δ in (2.1), for $\alpha = \frac{1}{2} < \frac{\pi}{6}$. So by Lemma 2.3, every ball $B(z, \varepsilon)$, for $z \in Y'$, is not completely δ-T-ε-V. Applying Lemma 2.7 we obtain that for at most $\frac{1}{4R^2}T^2$ elements $z \in Y'$ the ball $B(z, \varepsilon)$ is δ-F-ε-V. That leaves at least $\frac{1}{4R^2}T^2 = \frac{1}{2R^2}T^2 - \frac{1}{4R^2}T^2$ many elements z of Y' for which the ball $B(z, \varepsilon)$ is not completely δ-T-ε-V and not

δ-F-ε-V. Namely there are at least $\frac{1}{4R^2}T^2$ points, on the boundaries of these balls, which are not ε-visible. □

3. Proof of Theorem 1.3

Consider the set

$$Y \overset{\text{def}}{=} \{y_k \mid k \geq 2\} \subseteq \mathbb{C}, \tag{3.1a}$$

where

$$y_k = r_k e^{i\phi_k} \in \mathbb{C}, \qquad \text{and} \qquad \begin{cases} r_k &= k \log k \\ \phi_k &= \log^{1/2}(\log k) \end{cases} \tag{3.1b}$$

(with $\mathbb{R}^2$ and $\mathbb{C}$ being identified).

Theorem 1.3 is derived from the following proposition.

Proposition 3.1. *The set Y satisfies* $\mathbf{vis}(Y) = \varnothing$ *and*

$$\lim_{r\to\infty} \frac{G_Y(r)\log r}{r} = 1. \tag{3.2}$$

Proof. The growth rate (3.2) of Y is easily validated. In order to prove that $\mathbf{vis}(Y) = \varnothing$, we have to show that for any ray $L_{x,v} = \{x + tv \mid t \in [0, \infty)\}$ one has:

$$\mathbf{dist}(L_{x,v}, Y) = 0 \qquad (\forall x \in \mathbb{R}^2,\ \forall v \in \mathbb{S}^1). \tag{3.3}$$

Fix $x \in \mathbb{R}^2$ and $v \in \mathbb{S}^1$. Since the union $U = \bigcup_{k\geq 3} \overline{y_k, y_{k+1}}$ of the segments $\overline{y_k, y_{k+1}}$ forms an expanding spiral in $\mathbb{R}^2$ (spinning counter-clockwise), the set

$$K = K(L_{x,v}) := \{k \geq 3 \mid \overline{y_k, y_{k+1}} \cap L_{x,v} \neq \varnothing\} \tag{3.4}$$

is infinite. We shall prove that in fact

$$\lim_{\substack{k\to\infty \\ k\in K}} \mathbf{dist}(L_{x,v}, y_k) = 0. \tag{3.5}$$

This would imply (3.3) and complete the proof of Proposition 3.1.

Observe the following three estimates (see (3.1b)):

$$\phi'_k \overset{\text{def}}{=} \phi_{k+1} - \phi_k = O\Big(\tfrac{1}{k \cdot \log k \cdot \log^{1/2}(\log k)}\Big), \tag{3.6a}$$

$$r'_k \overset{\text{def}}{=} r_{k+1} - r_k = O(\log k) \tag{3.6b}$$

$$|y_{k+1} - y_k| = O(\log k). \tag{3.6c}$$

The first two, (3.6a) and (3.6b), are straightforward, and the third one easily follows:

$$\begin{aligned}|y_{k+1}-y_k| &= |r_{k+1}e^{i\phi_{k+1}} - r_k e^{i\phi_k}| \le |r_{k+1}e^{i\phi_{k+1}} - r_k e^{i\phi_{k+1}}| \\ &+ r_k|e^{i\phi_{k+1}} - e^{i\phi_k}| = |r'_k| + r_k\,|e^{i\phi'_k} - 1| = O(\log k) \\ &+ O(r_k\cdot\phi'_k) = O(\log k) + O\big(\tfrac{1}{\log^{1/2}(\log k)}\big) = O(\log k).\end{aligned}$$

For $x\in\mathbb{R}^2$, denote by $\phi'_{k,x}\in[0,\pi]$ the angle between the vectors $\overrightarrow{xy_k} = y_k - x$ and $\overrightarrow{xy_{k+1}} = y_{k+1}-x$.

Note that in view of (3.1b) and (3.6a), we have

$$\phi'_{k,\mathbf{0}} = \phi_{k+1}-\phi_k = O\big(\tfrac{1}{k\cdot\log k\cdot\log^{1/2}(\log k)}\big) \tag{3.7}$$

where $\mathbf{0}=(0,0)$. Denote by $S_{k,x}$ the area of the triangle $\triangle(x,y_k,y_{k+1})$, with vertices x, y_k, y_{k+1}. Then

$$S_{k,x} = \tfrac{1}{2}\,|y_k - x|\cdot|y_{k+1}-x|\cdot\sin\phi'_{k,x} \tag{3.8}$$

and hence, in view of (3.1b) and (3.7),

$$\begin{aligned}S_{k,\mathbf{0}} &= \tfrac{1}{2}\,|y_k|\cdot|y_{k+1}|\cdot\sin\phi'_{k,\mathbf{0}} \\ &= O(k^2\cdot\log^2 k\cdot\phi'_{k,\mathbf{0}}) = O\big(\tfrac{k\cdot\log k}{\log^{1/2}(\log k)}\big).\end{aligned} \tag{3.9}$$

Denote by $\Re(z), \Im(z)\in\mathbb{R}$ the real and imaginary parts of $z\in\mathbb{C}$.

Since $x\in\mathbb{C}=\mathbb{R}^2$ is fixed, the numbers $a=\Re(x)$, $b=\Im(x)$ are also fixed. Then

$$S_{k,x} = \frac{1}{2}\left|\det\begin{pmatrix}\Re(y_k)-a & \Re(y_{k+1}-y_k)\\ \Im(y_k)-b & \Im(y_{k+1}-y_k)\end{pmatrix}\right| \le \big|S^{(1)}_{k,x}\big| + \big|S^{(2)}_{k,x}\big|$$

where $S^{(1)}_{k,x} = \frac{1}{2}\det\begin{pmatrix}\Re(y_k) & \Re(y_{k+1}-y_k)\\ \Im(y_k) & \Im(y_{k+1}-y_k)\end{pmatrix}$ and $S^{(2)}_{k,x} = \det\begin{pmatrix}a & \Re(y_{k+1}-y_k)\\ b & \Im(y_{k+1}-y_k)\end{pmatrix}$.

In view of (3.9) and (3.6c), we have $\big|S^{(1)}_{k,x}\big| = S_{k,\mathbf{0}} = O\Big(\frac{k\cdot\log k}{\log^{1/2}(\log k)}\Big)$ and $\big|S^{(2)}_{k,x}\big| = O(|y_{k+1}-y_k|) = O(\log k)$ (as x is fixed). Thus

$$S_{k,x} \le \big|S^{(1)}_{k,x}\big| + \big|S^{(2)}_{k,x}\big| = O\big(\tfrac{k\cdot\log k}{\log^{1/2}(\log k)}\big).$$

Since $|y_k - x|^{-1} = O(k^{-1}\log^{-1}k)$, it follows from (3.8) that

$$\sin\phi'_{k,x} = O\big(|y_k-x|^{-1}\,|y_{k+1}-x|^{-1}\cdot S_{k,x}\big) = O\Big(\frac{1}{k\cdot\log k\cdot\log^{1/2}(\log k)}\Big). \tag{3.10}$$

Now assume that $k\in K$. Then the ray $L_{x,v}$ intersects the segment $[y_k,y_{k+1}]$. Let $\psi_{k,x}$ be the angle between the ray $\overrightarrow{x,y_k}$ and the ray $L_{x,v}$.

This angle forms a part of the angle between the vectors $\overrightarrow{xy_k}$ and $\overrightarrow{xy_{k+1}}$, hence

$$0 \le \psi_{k,x} \le \phi'_{k,x} < \pi/2 \qquad (k \in K).$$

Taking into account the estimate (3.10), we obtain

$$\begin{aligned}\mathbf{dist}(L_{x,v}, Y) &\le \mathbf{dist}(L_{x,v}, y_k) = |y_k| \sin \psi_{k,x} \le |y_k| \sin \phi'_{k,x} \\ &= (k \cdot \log k) \cdot O\Big(\tfrac{1}{k \cdot \log k \cdot \log^{1/2}(\log k)}\Big) = O\big(\tfrac{1}{\log^{1/2}(\log k)}\big) \quad (k \in K).\end{aligned}$$

This proves (3.3) and completes the proof of Proposition 3.1. □

4. Proof of Theorem 1.4

The proof of Theorem 1.4 is provided only for the case of $d = 2$. The general case is handled in a similar way.

The proof of Theorem 1.4, for $d = 2$, is partitioned into two parts. The implications (1)⇒(2) and (2)⇒(3) are established by Propositions 4.1 and 4.3, respectively.

Proposition 4.1. *Let $Y \subseteq \mathbb{R}^2$ be a discrete subset such that $G_Y(r) < \frac{r}{\log^{1+\varepsilon} r}$, for some $\varepsilon > 0$ and all large r. Then $\sum\limits_{y \in Y \setminus \{\mathbf{0}\}} \frac{1}{\|y\|} < \infty$.*

Proof. For $k \ge 1$, denote $Y_k = \{y \in Y \mid \|y\| \le 2^k\}$ and

$$Z_k = Y_{k+1} \setminus Y_k = \left\{ y \in Y \,\middle|\, 2^k < \|y\| \le 2^{k+1} \right\}.$$

Then, for large k, we have

$$\sum_{y \in Z_k} \frac{1}{\|y\|} \le |Z_k| \cdot 2^{-k} \le |Y_{k+1}| \cdot 2^{-k} \le \frac{2^{k+1}}{\log^{1+\varepsilon}(2^{k+1})} \cdot 2^{-k} = \frac{O(1)}{k^{1+\varepsilon}}.$$

It follows that

$$\sum_{y \in Y \setminus \{\mathbf{0}\}} \frac{1}{\|y\|} \le \sum_{y \in Y_1 \setminus \{\mathbf{0}\}} \frac{1}{\|y\|} + \sum_{k \ge 1} \left(\sum_{y \in Z_k} \frac{1}{\|y\|} \right) < \infty.$$

□

Lemma 4.2. *Let $Y \subseteq \mathbb{R}^2$ be a discrete subset such that $\sum\limits_{y \in Y \setminus \{\mathbf{0}\}} \frac{1}{\|y\|} < \infty$ holds. Then, for Lebesgue almost all directions $v \in \mathbb{S}^1 \subseteq \mathbb{R}^2$, we have $\mathbf{0} \in \mathbf{vis}(Y, v)$. In particular, $\mathbf{0} \in \mathbf{vis}(Y)$.*

Proof. Recall that $L_{\mathbf{0},v} = \{vt \mid t \in [0,\infty)\} \subseteq \mathbb{R}^2$. Let $Y' = Y \setminus \{\mathbf{0}\}$ and $\varepsilon > 0$. Set

$$\begin{aligned} D_Y(\varepsilon) &\stackrel{\text{def}}{=} \{v \in \mathbb{S}^1 \mid \mathbf{dist}(L_{\mathbf{0},v}, Y') < \varepsilon\} \\ &= \bigcup_{y \in Y'} \{v \in \mathbb{S}^1 \,|\, \mathbf{dist}(L_{\mathbf{0},v}, y) < \varepsilon\}. \end{aligned}$$

Then

$$\lambda(D_Y(\varepsilon)) \le \sum_{y \in Y'} \lambda(\{v \in \mathbb{S}^1 \mid \mathbf{dist}(L_{\mathbf{0},v}, y) < \varepsilon\}) \tag{4.1}$$

where λ stands for the Lebesgue measure on the unit circle $\mathbb{S}^1 \subseteq \mathbb{R}^2$, $\lambda(\mathbb{S}^1) = 2\pi$.

Now assume that $0 < \varepsilon < \min\limits_{y \in Y'} \|y\|$. Then one verifies that

$$\lambda(\{v \in \mathbb{S}^1 \mid \mathbf{dist}(L_{\mathbf{0},v}, y) < \varepsilon\}) = 2 \arcsin \tfrac{\varepsilon}{\|y\|} < \tfrac{\pi\varepsilon}{\|y\|},$$

for every $y \in Y'$ (the inequality $2 \arcsin t < \pi t$, for $0 < t < 1$, is used).

By substituting the last inequality into (4.1), we derive that $\lambda(D_Y(\varepsilon)) \le \pi\varepsilon\, c$, where $c = c(Y) = \sum\limits_{y \in Y'} \dfrac{1}{\|y\|} < \infty$.

Next consider the set $D_Y = \{v \in \mathbb{S}^1 \mid \mathbf{dist}(L_{\mathbf{0},v}, Y') = 0\}$. Since for every $\varepsilon > 0$ we have $D_Y \subseteq D_Y(\varepsilon)$, and since $\lim\limits_{\varepsilon \to 0+} \lambda(D_Y(\varepsilon)) = 0$, we conclude that $\lambda(D_Y) = 0$, and hence

$$\lambda(\mathbb{S}^1 \setminus D_Y) = \lambda\{v \in \mathbb{S}^1 \mid \mathbf{dist}(L_{\mathbf{0},v}, Y') > 0\} = 2\pi.$$

□

Proposition 4.3. *Let $Y \subseteq \mathbb{R}^2$ be a discrete subset and let $x \in \mathbb{R}^2$ be an arbitrary point. Assume that $\sum\limits_{y \in Y \setminus \{\mathbf{0}\}} \frac{1}{\|y\|} < \infty$ holds. Then, for Lebesgue almost all directions $v \in \mathbb{S}^1$, we have $x \in \mathbf{vis}(Y, v)$.*

Proof. Let $Z = Y - x = \{y - x \mid y \in Y\}$. Observe the implication

$$\sum_{y \in Y \setminus \{\mathbf{0}\}} \tfrac{1}{\|y\|} < \infty \quad \Longrightarrow \quad \sum_{z \in Z \setminus \{\mathbf{0}\}} \tfrac{1}{\|z\|} < \infty.$$

By Lemma 4.2, for Lebesgue almost all directions $v \in \mathbb{S}^1$, we have $\mathbf{0} \in \mathbf{vis}(Z, v)$; hence $x \in \mathbf{vis}(Z + x, v) = \mathbf{vis}(Y, v)$. □

5. Proof of Theorem 1.5

In Theorem 1.5 we construct a large (density 1 and relatively dense) subset $Y \subseteq \mathbb{Z}^d$ with no hidden points for Y.

Proof of Theorem 1.5. For simplicity, the construction is presented only for dimension $d = 2$. The same idea works for general $d \geq 2$.

Outline of the construction. We start with arbitrary ordering of the set $\mathbb{Z}^2$ in a sequence $(z_k)_{k\geq 1}$.

Then, we inductively construct an increasing sequence $(m_k)_{k\geq 1}$ of positive integers (the details are below, following (5.4)).

Given z_k and m_k, the vectors $v_k \in \mathbb{Z}^2$ and the sets $Y_k \subseteq \mathbb{Z}^2$ are determined as follows:

$$v_k \stackrel{\text{def}}{=} (m_k, 1) \in \mathbb{Z}^2; \tag{5.1}$$

$$Y_k \stackrel{\text{def}}{=} \{z_k + nv_k \mid n \in \mathbb{N} \setminus \{0\}\} \subseteq \mathbb{Z}^2. \tag{5.2}$$

Finally, we define set Y by setting

$$\tilde{Y} \stackrel{\text{def}}{=} \bigcup_{k\geq 1} Y_k; \quad Y \stackrel{\text{def}}{=} \mathbb{Z}^2 \setminus \tilde{Y}. \tag{5.3}$$

We claim that every point $z \in \mathbb{R}^2$ is visible for Y, i. e. condition (1) of Theorem 1.5 is satisfied (regardless of the choice of integers m_k).

Indeed, if $z \notin \mathbb{Z}^2$, the claim is obvious (z is visible in either a horizontal or a vertical direction). Otherwise $z = z_k$ for some $k \geq 1$, and, since $Y \subseteq \mathbb{Z}^2 \setminus Y_k$, we get

$$z = z_k \in \mathbf{vis}(\mathbb{Z}^2 \setminus Y_k, v_k) \subseteq \mathbf{vis}(\mathbb{Z}^2 \setminus Y_k) \subseteq \mathbf{vis}(Y). \tag{5.4}$$

Construction of a sequence (m_k). We describe an inductive procedure for selecting integers m_k to assure that conditions (2), (3) and (4) of the theorem are met.

Let $m_1 > \max\{M, 4/\varepsilon, 2\|z_1\|\}$ and proceeds by induction.

Assume that a strictly increasing K terms long sequence of numbers $(m_k)_{k=1}^K$ has been already selected, $K \geq 1$. Then the vectors v_k and the sets Y_k are determined by (5.1) and (5.2). One easily verifies that for each $k = 1, \ldots, K$

$$\lim_{m\to\infty} \mathbf{dist}(Y_k, \{z_{k+1} + t(m, 1) \mid t \in [1, \infty)\} = \infty. \tag{5.5}$$

We select m_{K+1} large enough to satisfy the inequalities

$$m_{K+1} > \max\{2^{K+1}/\varepsilon, 2\|z_{K+1}\|, m_K\} \tag{5.6a}$$

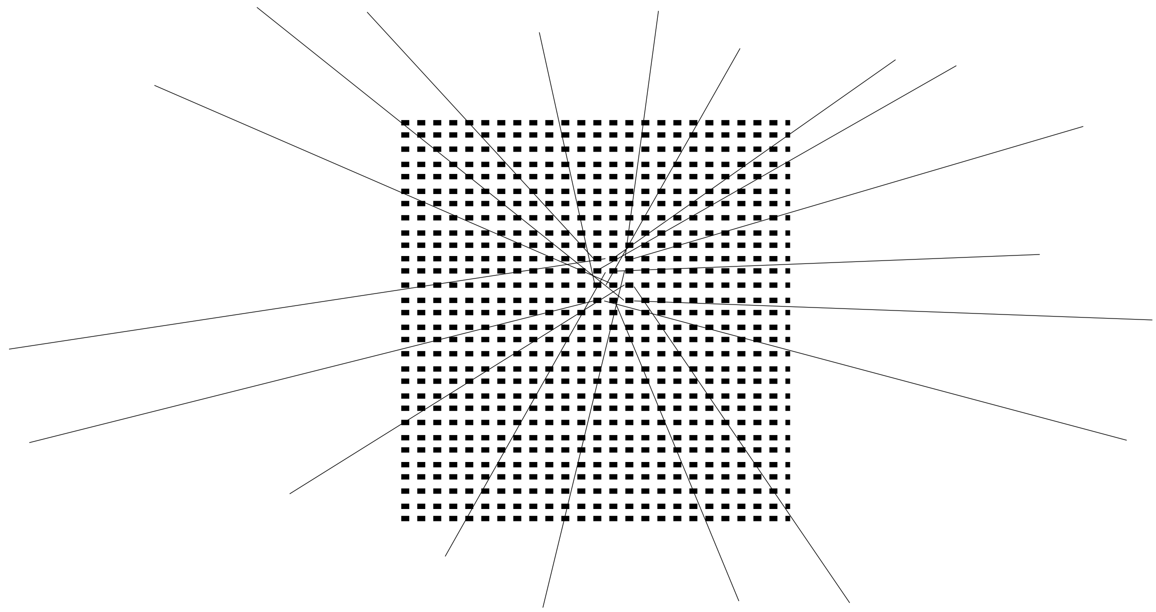

FIGURE 5. For each point z_k, a line of vision for z_k is created by removing all the integer points on a particular ray, which is initiated in z_k.

and

$$\mathbf{dist}(Y_k, Y_{K+1}) > M \quad (1 \leq k \leq K) \tag{5.6b}$$

where $Y_{K+1} = \{z_{K+1} + n(m_{K+1}, 1) \mid n \geq 1\}$ is set in accordance with (5.2) and (5.1) (note that the inequality (5.6b) can be achieved because of (5.5)).

This completes the inductive construction of the sequence (m_k).

Validation of condition (2). Note that $\|v_k\| = \|(m_k, 1)\| > m_k > 2\|z_k\|$ for all $k \geq 1$ (see (5.1) and (5.6a)). It follows that, for all $n, k \in \mathbb{N} \setminus \{0\}$,

$$\|nv_k + z_k\| \geq n\|v_k\| - \|z_k\| > (n - \tfrac{1}{2})\|v_k\| \geq \tfrac{n}{2}\|v_k\| > \tfrac{nm_k}{2}.$$

For any $k \geq 1$, in view of the definition of Y_k (see (5.2)), we obtain

$$G_{Y_k}(r) = \#\{n \geq 1 \mid \|nv_k + z_k\| < r\} \leq$$
$$\leq \#\{n \geq 1 \mid \tfrac{nm_k}{2} < r\} < \tfrac{2r}{m_k} < 2r\varepsilon 2^{-(k+1)} = r\varepsilon 2^{-k},$$

and, since $\tilde{Y} = \bigcup_k Y_k$ (see (5.3)), we conclude that

$$G_{\tilde{Y}}(r) \le \sum_{k\ge 1} G_{Y_k}(r) < r\varepsilon \sum_{k\ge 1} 2^{-k} = r\varepsilon,$$

validating condition (2) of Theorem 1.5.

Validation of conditions (3) and (4). To validate condition (4), we have to establish the implication $(y_1, y_2 \in \tilde{Y},\ y_1 \ne y_2) \implies \mathbf{dist}(y_1, y_2) \ge M$.

Since $y_1, y_2 \in \tilde{Y} = \bigcup_k Y_k$, there are $k_1, k_2 \in \mathbb{N}$ such that $y_1 \in Y_{k_1}$, $y_2 \in Y_{k_2}$. If $k_1 = k_2$, set $k = k_1$; then $y_1, y_2 \in Y_k$ and (since $y_1 \ne y_2$) we obtain $\mathbf{dist}(y_1, y_2) \ge \|v_k\| > m_k > M$. On the other hand, if $k_1 \ne k_2$, we may assume that $k_1 > k_2$, and then $\mathbf{dist}(y_1, y_2) \ge \mathbf{dist}(Y_{k_1}, Y_{k_2}) > M$ (see (5.6b)). This validates condition (4).

In order to validate condition (3), we show that every ball B of radius $\sqrt{2}$ contains a point in Y (this claim holds only for $d = 2$; for $d \ge 3$, the required radius could be taken $\sqrt{d}$). Let $B = B(z, \sqrt{2})$ where $z = (a, b) \in \mathbb{R}^2$. Then both points $y_1 = (\lfloor a \rfloor, \lfloor b \rfloor)$ and $y_2 = (\lfloor a \rfloor + 1, \lfloor b \rfloor)$ lie in $B \cap \mathbb{Z}^2$. Since $\|y_1 - y_2\| = 1 < M$, we have $y_i \notin \tilde{Y}$ for at least one $i \in \{1, 2\}$ (due to the already established condition (4)). Then $y_i \in Y$ (see (5.3)), and hence $y_i \in B \cap Y$. Thus $B \cap Y \ne \emptyset$. The proof of Theorem 1.5 is complete.

□

6. Proof of Theorem 1.7

We begin with the following lemma, which is the key for the proof of Theorem 1.7.

Lemma 6.1. *Let $Y \subseteq \mathbb{R}^2$ be a discrete set such that $\lim_{r\to\infty} \frac{r}{G_Y(r)} = 0$. Let a non-zero vector $\mathbf{0} \ne v \in \mathbb{R}^2$ and an $\varepsilon > 0$ be given. Then:*

(A) *For almost every $z \in Y$ (in the sense of (1.5)), one can find a point $w \in Y \setminus \{z\}$ and an integer $k \ge 0$ such that*

$$\|(z - w) - kv\| < \varepsilon. \tag{6.1}$$

(B) *Assume in addition that Y is uniformly separated. Then for every integer $M \ge 1$ and for almost every $z \in Y$ there exists $w \in Y \setminus \{z\}$ and an integer $k \ge M$ such that (6.1) holds.*

Proof of (A) in Lemma 6.1. Without loss of generality we may assume that $v = (1, 0)$.

Fix $r > 1$. Divide the interval $[-r, r)$ into $N_1 = \lceil 4r/\varepsilon \rceil$ half-closed intervals S_i of equal length $d_1 = \frac{2r}{N_1} \le \varepsilon/2$. Divide the interval $[0, 1)$ into $N_2 = \lceil 2/\varepsilon \rceil$ half-closed intervals I_j of equal length $d_2 = \frac{1}{N_2} \le \varepsilon/2$.

For $z = (z_1, z_2) \in \mathbb{R}^2$, denote by $\pi_j(z) = z_j \in \mathbb{R}$, $j = 1, 2$. For $1 \le i \le N_1$ and $1 \le j \le N_2$, set

$$Y_{i,j}(r) \stackrel{\text{def}}{=} \{y \in Y \cap B(\mathbf{0}, r) \mid \pi_2(y) \in S_i \ \text{ and } \ \{\pi_1(y)\} \in I_j\} \tag{6.2}$$

where $\{\pi_1(y)\} \in [0, 1)$ stands for the fractional part of $\pi_1(y)$.

Next we prove the following implication:

$$z, w \in Y_{i,j}(r) \implies \|(z - w) - kv\| < \varepsilon, \tag{6.3}$$

where $k = \lfloor \pi_1(z) \rfloor - \lfloor \pi_1(w) \rfloor \in \mathbb{Z}$.

Indeed, if $z, w \in Y_{i,j}(r)$ for some $1 \le i \le N_1$, $1 \le j \le N_2$, then

$$|\{\pi_1(z)\} - \{\pi_1(w)\}| < |I_j| = d_2 \le \varepsilon/2 \tag{6.4}$$

and hence

$$|(\pi_1(z) - \pi_1(w)) - k| < \varepsilon/2.$$

We also have

$$|\pi_2(z) - \pi_2(w)| < |S_i| = d_1 \le \varepsilon/2, \tag{6.5}$$

and hence

$$\|(z - w) - kv\| = \left\|\big(\pi_1(z) - \pi_1(w) - k,\ \pi_2(z) - \pi_2(w)\big)\right\| < \sqrt{2}\cdot\varepsilon/2 < \varepsilon,$$

completing the proof of the implication (6.3).

Denote by Y' the set of $z \in Y$ such that for every $w \in Y \setminus \{z\}$ and every integer $k \ge 0$ the inequality

$$\|(z - w) - kv\| \ge \varepsilon \tag{6.6}$$

holds. By (6.3)

$$\#(Y_{i,j}(r) \cap Y') \le 1, \quad \text{for all } 1 \le i \le N_1, 1 \le j \le N_2. \tag{6.7}$$

Since $B(\mathbf{0}, r) \cap Y' = \bigcup_{i,j} (Y_{i,j}(r) \cap Y')$, we conclude that

$$\#(B(\mathbf{0}, r) \cap Y') \le \#((i, j)) = N_1 N_2 = \lceil 4r/\varepsilon \rceil \cdot \lceil 2/\varepsilon \rceil .$$

Therefore $\limsup_{r\to\infty} \frac{\#(B(\mathbf{0},r)\cap Y')}{r} \le \lceil 4/\varepsilon \rceil \cdot \lceil 2/\varepsilon \rceil$. Finally, since $\lim_{r\to\infty} \frac{r}{G_Y(r)} = 0$, we derive that $\lim_{r\to\infty} \frac{\#(B(\mathbf{0},r)\cap Y')}{G_Y(r)} = 0$, completing the proof of (A) in Lemma 6.1. □

Proof of (B) in Lemma 6.1. As in the proof of (A), without loss of generality we assume that $v = (0, 1)$. Since Y is uniformly separated, there exists a $\delta > 0$ such that $\|y_1 - y_2\| \geq \delta$, for all distinct $y_1, y_2 \in Y$.

We assume that $\varepsilon < \delta < 1$. Fix $r > 1$. Define the integers N_1, N_2, the intervals S_i, I_j, the numbers d_1, d_2 and the sets $Y_{i,j}(r)$ (for $1 \leq i \leq N_1, 1 \leq j \leq N_2$) as in the proof of (A) (see (6.2)). We claim that for distinct $y_1, y_2 \in Y_{i,j}(r)$ we have

$$|\pi_1(y_1) - \pi_1(y_2)| > \delta/2. \tag{6.8}$$

Indeed,

$$(\pi_1(y_1) - \pi_1(y_2))^2 + (\pi_2(y_1) - \pi_2(y_2))^2 = \|y_2 - y_1\|^2 > \delta^2$$

and, since $\varepsilon < \delta$ and $|\pi_2(y_1) - \pi_2(y_2)| < \varepsilon/2$ (see (6.5)), we get

$$(\pi_1(y_1) - \pi_1(y_2))^2 > \delta^2 - \varepsilon^2/4 > \delta^2 - \delta^2/4 > \delta^2/4,$$

and (6.8) follows. Denote by Y'_M the set of $z \in Y$ such that for every $w \in Y \setminus \{z\}$ and every $k \geq M$ the inequality

$$\|(z - w) - kv\| \geq \varepsilon \tag{6.9}$$

holds. Let $N = \lceil 2M/\delta \rceil$. We claim that

$$\#(Y_{i,j}(r) \cap Y'_M) \leq N, \quad \text{for all } 1 \leq i \leq N_1, 1 \leq j \leq N_2. \tag{6.10}$$

That is, no set $Y_{i,j}(r) \cap Y'_M$ contains more than N elements.

Assume to the contrary that, for some choice of i, j, we have $N + 1$ distinct elements $y_1, y_2, \ldots, y_{N+1}$ lying in the same set $Y_{i,j}(r) \cap Y'_M$. We may assume that these $N+1$ elements are arranged in such a way that $\pi_1(y_{p+1}) - \pi_1(y_p) > \delta/2$, for all $p = 1, 2, \ldots, N$ (see (6.8)). Then

$$\pi_1(y_{N+1}) - \pi_1(y_1) > N \cdot \delta/2 \geq M.$$

In view of (6.3), we obtain $\|(y_{N+1} - y_1) - kv\| < \varepsilon$ where

$$k = \lfloor \pi_1(y_{N+1}) \rfloor - \lfloor \pi_1(y_1) \rfloor \geq \lfloor \pi_1(y_{N+1}) - \pi_1(y_1) \rfloor \geq M.$$

This contradicts the assumption that $y_{N+1} \in Y'_M$, proving (6.10).

Since $B(\mathbf{0}, r) \cap Y'_M = \bigcup_{i,j} (Y_{i,j}(r) \cap Y'_M)$, we conclude that

$$\#(B(\mathbf{0}, r) \cap Y'_M) \leq N \cdot \#((i,j)) = N N_1 N_2 = \lceil 2M/\delta \rceil \cdot \lceil 4r/\varepsilon \rceil \cdot \lceil 2/\varepsilon \rceil,$$

and hence $\limsup_{r\to\infty} \frac{\#(B(\mathbf{0},r)\cap Y'_M)}{r} \leq \lceil 2M/\delta \rceil \cdot \lceil 4r/\varepsilon \rceil \cdot \lceil 2/\varepsilon \rceil$. Since $\lim_{r\to\infty} \frac{r}{G_Y(r)} = 0$, we derive $\lim_{r\to\infty} \frac{\#(B(\mathbf{0},r)\cap Y'_M)}{G_Y(r)} = 0$, completing the proof of (B) in Lemma 6.1. □

We completed the proofs of both parts of Lemma 6.1. Part (B) of this lemma, with $M = 1$, is used in the proof of Theorem 1.7.

Proof of Theorem 1.7. The proof is by induction. Assume the assertion for every tree with less than $m+1$ vertices, and let Γ be a tree with $m+1$ vertices $V = \{x_0, \ldots, x_m\}$, embedded in the plane. Let $\Gamma' \stackrel{\text{def}}{=} \Gamma \setminus \{x_0\}$ (the graph obtained from Γ by removing x_0 and all its adjacent edges). Let c be the number of connected components of Γ', then Γ' is a disjoint union of c trees $\Gamma_1, \ldots, \Gamma_c$, each with less than $m+1$ vertices. Denote by $x_{i_j} \in \Gamma_j$ the unique neighbor of x_0 in Γ_j, then for every $j \in \{1, \ldots, c\}$, by the induction hypothesis, for almost every $y \in Y$, (Γ_j, x_{i_j}) can be ε-realized from y in Y.

Let $Y_j \subseteq Y$ be the set of points $y \in Y$ for which (Γ_j, x_{i_j}) cannot be ε-realized from y. Then $Y' \stackrel{\text{def}}{=} Y \backslash (Y_1 \cup \ldots \cup Y_c)$ still satisfies $\lim\limits_{r \to \infty} \frac{r}{G_{Y'}(r)} = 0$, and, for every $y \in Y'$ and every $j \in \{1, \ldots, c\}$, the planar tree (Γ_j, x_{i_j}) can be ε-realized from y in Y. For each j consider the edge $\{x_0, x_{i_j}\}$ of Γ. By Lemma 6.1 (part B), for almost every $y \in Y'$ there exists a positive integer k_{i_j} and a point $z_{i_j} \in Y'$ such that

$$\left\| (y - z_{i_j}) - k_{i_j}(x_0 - x_{i_j}) \right\| < \varepsilon.$$

Hence for almost every $y \in Y'$ there exist positive integers $k_{i_1} \ldots, k_{i_c}$ and points $z_{i_1}, \ldots, z_{i_c} \in Y'$ such that for every $j \in \{1, \ldots, c\}$ we have

$$\left\| (y - z_{i_j}) - k_{i_j}(x_0 - x_{i_j}) \right\| < \varepsilon,$$

and the assertion follows. $\square$

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